Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

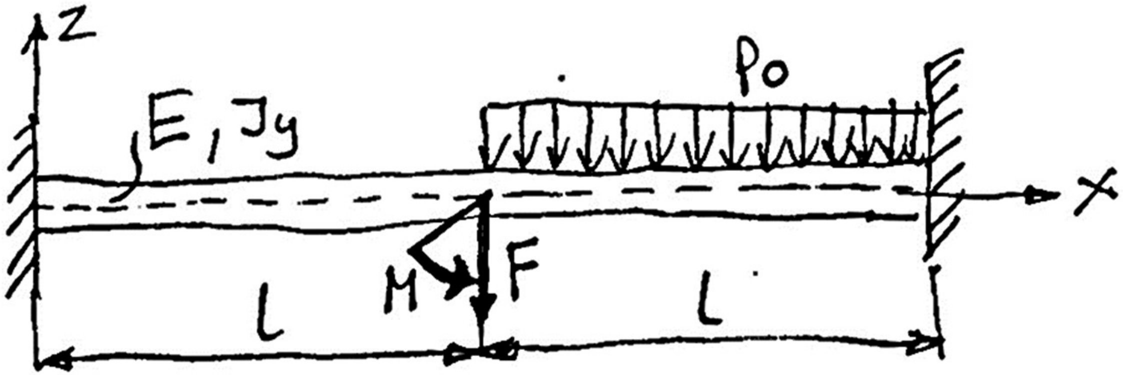
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Example. FE model of a beam

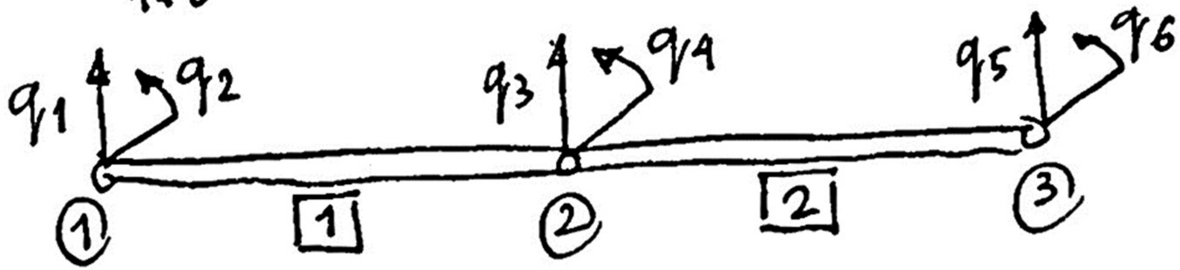
05.2021

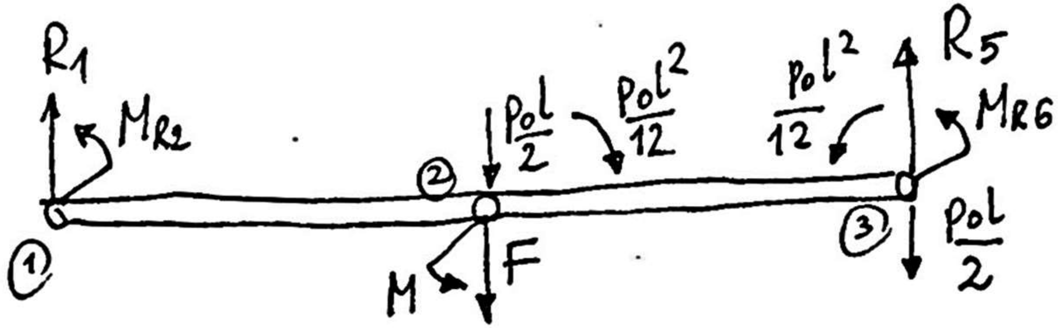
EXAMPLE. WRITE A SET OF FE EQUATIONS. FIND NODAL DISPLACEMENTS, REACTIONS AND BENDING MOMENT DISTRIBUTION. USE TWO BEAM ELEMENTS.



NODAL PARAMETERS

$$\lfloor q \rfloor_{1\times 6} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6 \rfloor$$





NODAL LOAD

$$\underset{1\times 6}{\lfloor F \rfloor}^n = \lfloor R_1, M_{R2}, -F, M, R_5, M_{R6} \rfloor$$

EQUIVALENT LOAD

$$\underset{1\times 6}{\lfloor F \rfloor}^e = \underset{1\times 6}{\lfloor F \rfloor}_1^* + \underset{1\times 6}{\lfloor F \rfloor}_2^* = \lfloor \overbrace{F_{11}, F_{21}, F_{31}, F_{41}, 0, 0}^{=0} \rfloor +$$

$$+ \lfloor 0, 0, F_{12}, F_{22}, F_{32}, F_{42} \rfloor =$$

$$= \lfloor 0+0, 0+0, 0-\frac{p_0 l}{2}, 0-\frac{p_0 l^2}{12}, 0-\frac{p_0 l}{2}, 0+\frac{p_0 l^2}{12} \rfloor$$

GLOBAL LOAD

$$\underset{1\times 6}{\lfloor F \rfloor} = \underset{1\times 6}{\lfloor F \rfloor}^n + \underset{1\times 6}{\lfloor F \rfloor}^e =$$

$$= \lfloor R_1, M_{R2}, -F-\frac{p_0 l}{2}, M-\frac{p_0 l^2}{12}, R_5-\frac{p_0 l}{2}, M_{R6}+\frac{p_0 l^2}{12} \rfloor =$$

$$= \lfloor F_1, F_2, F_3, F_4, F_5, F_6 \rfloor$$

(N) (Nm) ...

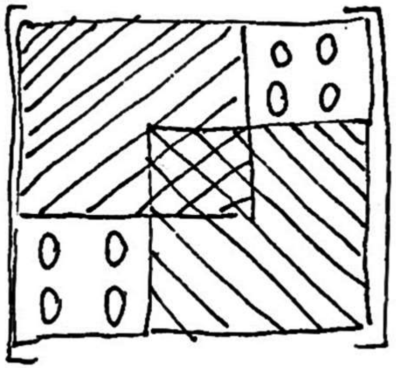
STIFFNESS MATRICES

$$\underset{4\times4}{[k]_1} = \underset{4\times4}{[k]_2} = \frac{2EJ_y}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

$$\underset{6\times6}{[k]_1^*} = \left[\begin{array}{c|cc} \underset{4\times4}{[k]_1} & \begin{matrix}0&0\\0&0\\0&0\\0&0\end{matrix} \\ \hline \begin{matrix}0&0&0&0\\0&0&0&0\end{matrix} & \begin{matrix}0&0\\0&0\end{matrix} \end{array}\right] ; \quad \underset{6\times6}{[k]_2^*} = \left[\begin{array}{cc|c} \begin{matrix}0&0\\0&0\end{matrix} & \begin{matrix}0&0&0&0\\0&0&0&0\end{matrix} \\ \hline \begin{matrix}0&0\\0&0\\0&0\\0&0\end{matrix} & [k]_2 \end{array}\right]$$

GLOBAL STIFFNESS MATRIX

$$\underset{6\times6}{[K]} = \underset{6\times6}{[k]_1^*} + \underset{6\times6}{[k]_2^*} =$$



SET OF EQUATIONS + BOUNDARY CONDITIONS

$$\underset{6\times6}{[K]} \cdot \underset{6\times1}{\{q\}} = \underset{6\times1}{\{F\}} \; + \; \text{B.C.:} \; q_1 = 0, \; q_2 = 0, \; q_5 = 0, \; q_6 = 0$$

SOLUTION

$$\cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{2EJ_y}{l^3} \begin{bmatrix} 6+6 & -3l+3l \\ -3l+3l & 2l^2+2l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{EJ_y}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

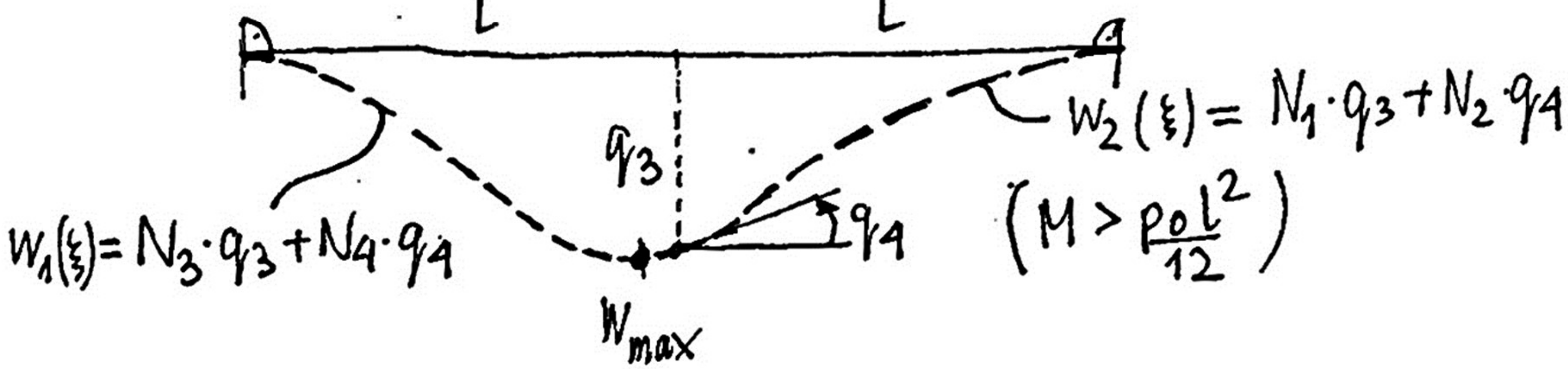
$$\frac{24EJ_y}{l^3} \cdot q_3 = F_3 \implies q_3 = -\frac{\left(F + \frac{p_0 l}{2}\right) l^3}{24EJ_y} \quad (mm)$$

$$\frac{8EJ_y}{l} q_4 = F_4 \implies q_4 = \frac{\left(M - \frac{p_0 l^2}{12}\right) l}{8EJ_y} \quad (rad)$$

BEAM DEFLECTION

$$W_1(\xi) = \lfloor N \rfloor \cdot \{q\}_1 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_1$$

$$W_2(\xi) = \lfloor N \rfloor \cdot \{q\}_2 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2$$



REACTIONS

$$\frac{2EJ_y}{l^3}\left\lfloor 6,\ 3l,\ -6,\ 3l,\ 0,\ 0\right\rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = F_1 = R_1$$

$$R_1 = -\frac{12EJ_y}{l^3}\cdot\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right) + \frac{6EJ_y}{l^2}\cdot\frac{\left(M-\frac{p_0 l^2}{12}\right)l}{8EJ_y} =$$

$$= \frac{1}{2}\left(F+\frac{p_0 l}{2}\right) + \frac{3}{4l}\left(M-\frac{p_0 l^2}{12}\right) =$$

$$= \underline{\frac{1}{2}F + \frac{3}{4l}\cdot M + \frac{3}{16}p_0 l}$$

$$\frac{2EJ_y}{l^3}\left[3l,\ 2l^2,\ -3l,\ l^2,\ 0,\ 0\right]\cdot\begin{Bmatrix}0\\0\\q_3\\q_4\\0\\0\end{Bmatrix} = F_2 = M_{e2}$$

$$M_{e2} = -\frac{6EJ_y}{l^2}\cdot\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right)+\frac{2EJ_y}{l}\cdot\frac{\left(M-\frac{p_0 l^2}{12}\right)l}{8EJ_y} =$$

$$= +\frac{1}{4}\left(F+\frac{p_0 l}{2}\right)l+\frac{1}{4}\left(M-\frac{p_0 l^2}{12}\right) = \underbrace{\frac{1}{4}Fl+\frac{1}{4}M+\frac{5}{48}p_0 l^2}$$

$$\frac{2EJ_y}{l^3} \lfloor 0,\ 0,\ -6,\ -3l,\ 6,\ -3l \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_5 = R_5 - \frac{p_o l}{2}$$

$$R_5 = \frac{-12\,EJ_y}{l^3} \cdot \left(- \frac{\left(F + \frac{p_o l}{2}\right) l^3}{24 EJ_y} \right) - \frac{6EJ_y}{l^2}\,\frac{\left(M - \frac{p_o l^2}{12}\right) l}{8 EJ_y} + \frac{p_o l}{2} =$$

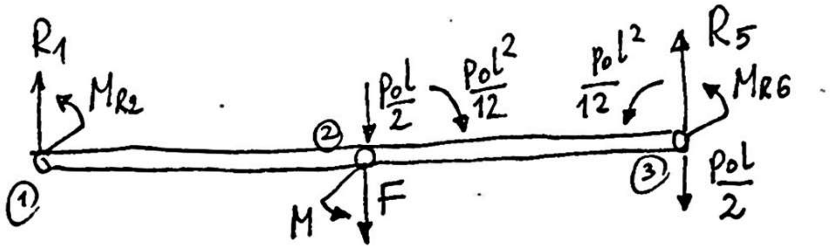
$$= \frac{1}{2}\left(F + \frac{p_o l}{2}\right) - \frac{3}{4l}\left(M - \frac{p_o l^2}{12}\right) + \frac{p_o l}{2} = \underline{\frac{1}{2}F - \frac{3}{4l} \cdot M + \frac{13}{16} p_o l}$$

$$\frac{2EJ_y}{l^3}\lfloor 0, 0, 3l, l^2, -3l, 2l^2 \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_6 = M_{R6} + \frac{p_0 l^2}{12}$$

$$M_{R6} = \frac{6EJ_y}{l^2}\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right) + \frac{2EJ_y}{l} \cdot \frac{\left(M - \frac{p_0 l^2}{12}\right)l}{8EJ_y} - \frac{p_0 l^2}{12} =$$

$$= -\frac{1}{4}\left(F + \frac{p_0 l}{2}\right)l + \frac{1}{4}\left(M - \frac{p_0 l^2}{12}\right) - \frac{p_0 l^2}{12} = \underline{-\frac{1}{4}Fl + \frac{1}{4}M - \frac{11}{48}p_0 l^2}$$

EQUILIBRIUM CHECK



$$\sum F_z = 0$$

$$R_1 - \frac{p_0 l}{2} - F - \frac{p_0 l}{2} + R_5 = 0$$

$$\frac{1}{2}F + \frac{3}{4l}\cdot M + \frac{3}{16}p_0 l - \frac{p_0 l}{2} - F - \frac{p_0 l}{2} + \frac{1}{2}F - \frac{3}{4l}\cdot M + \frac{13}{16}p_0 l = \underline{0}$$

$$\sum M_y^{x=0} = 0 \qquad +\circlearrowright$$

$$M_{R2} - \frac{p_0 l}{2}\cdot l + M - F\cdot l - \frac{p_0 l^2}{12} + \frac{p_0 l^2}{12} + R_5\cdot 2l + M_{R6} - \frac{p_0 l}{2}\cdot 2l = 0$$

$$M_{R2} - \frac{3}{2}p_0 l^2 + M - F\cdot l + R_5\cdot 2l + M_{R6} =$$

$$= \frac{1}{4}Fl + \frac{1}{4}M + \frac{5}{48}p_0 l^2 - \frac{3}{2}p_0 l^2 + M - F\cdot l + F\cdot l - \frac{3}{2}M + \frac{13}{8}p_0 l^2 +$$

$$- \frac{1}{4}Fl + \frac{1}{4}M - \frac{11}{48}p_0 l^2 = \frac{5}{48}p_0 l^2 - \frac{3}{2}p_0 l^2 + \frac{13}{8}p_0 l^2 - \frac{11}{48}p_0 l^2 =$$

$$= \frac{5 - 3\cdot 24 + 13\cdot 6 - 11}{48}p_0 l^2 = \frac{5 - 72 + 78 - 11}{48}p_0 l^2 = \underline{0}$$

BENDING MOMENT

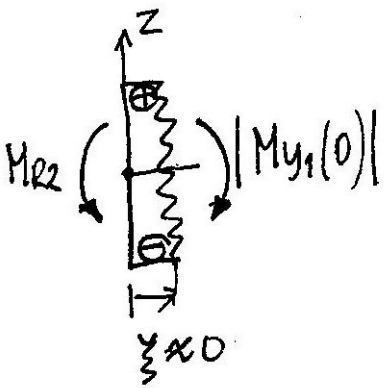
$$\boxed{1}: \quad M_{y1}(\xi) = EJ_y\, w_1'' = EJ_y \lfloor N'' \rfloor_{1\times 4} \cdot \{q\}_{4\times 1,\,1} =$$

$$= EJ_y \lfloor N_1'', N_2'', N_3'', N_4'' \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_1 = EJ_y \left(N_3''\cdot q_3 + N_4''\cdot q_4 \right) =$$

$$= EJ_y \left(\left(\frac{6}{l^2} - \frac{12}{l^3}\xi \right) q_3 + \left(-\frac{2}{l} + \frac{6}{l^2}\cdot \xi \right)\cdot q_4 \right)$$

$$M_{y1}(0) = EJ_y \left(\frac{6q_3}{l^2} - \frac{2q_4}{l} \right) = -\frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} - \frac{2\left(M - \frac{p_0 l^2}{12}\right) l}{8 l} =$$

$$= -\frac{1}{4}\left(Fl + \frac{5}{12} p_0 l^2 + M \right) \qquad (= -M_{R2})$$



$$M_{y1}(l) = EJ_y \left(\left(\frac{6}{l^2} - \frac{12}{l^2} \right)\cdot q_3 + \left(-\frac{2}{l} + \frac{6}{l} \right) q_4 \right) = EJ_y \left(-\frac{6}{l^2}\cdot q_3 + \frac{4}{l} q_4 \right) =$$

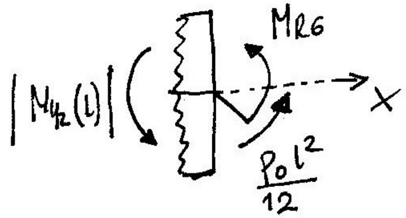
$$= \frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} + \frac{4\left(M - \frac{p_0 l^2}{12}\right) l}{8 l} = \frac{1}{4}\left(F\cdot l + \frac{1}{3} p_0 l^2 + 2M \right)$$

$$\boxed{2} : \quad M_{y_2}(\xi) = EJ_y\, w_2'' = EJ_y \lfloor N_1'', N_2'', N_3'', N_4'' \rfloor \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2 =$$

$$= EJ_y \left(N_1'' \cdot q_3 + N_2'' \cdot q_4 \right) = EJ_y \left(\left(-\frac{6}{l^2} + \frac{12}{l^3}\xi \right) q_3 + \left(-\frac{4}{l} + \frac{6}{l^2}\xi \right) \cdot q_4 \right)$$

$$M_{y_2}(0) = EJ_y \left(-\frac{6}{l^2} q_3 - \frac{4}{l} q_4 \right) = \frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} - \frac{4\left(M - \frac{p_0 l^2}{12}\right) l}{8l} =$$

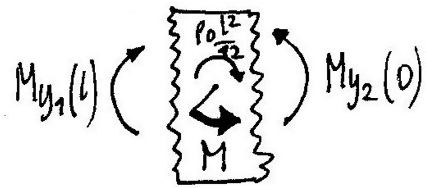
$$= \frac{1}{4}\left(Fl + \frac{2}{3} p_0 l^2 - 2M \right)$$



$$M_{y_2}(l) = EJ_y \left(\frac{6}{l^2} q_3 + \frac{2}{l} q_4 \right) = -\frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} + \frac{2\left(M - \frac{p_0 l^2}{12}\right) l}{8l} =$$

$$= -\frac{1}{4}\left(F \cdot l + \frac{7}{12} p_0 l^2 - M \right)$$

$$\left(= M_{R6} + \frac{p_0 l^2}{12} \right)$$



$$M_{y_1}(l) - M_{y_2}(0) = \frac{1}{4}\left(F \cdot l + \frac{1}{3} p_0 l^2 + 2M \right) - \frac{1}{4}\left(F \cdot l + \frac{2}{3} p_0 l^2 - 2M \right) =$$

$$= M - \frac{p_0 l^2}{12}$$

LETS ASSUME:

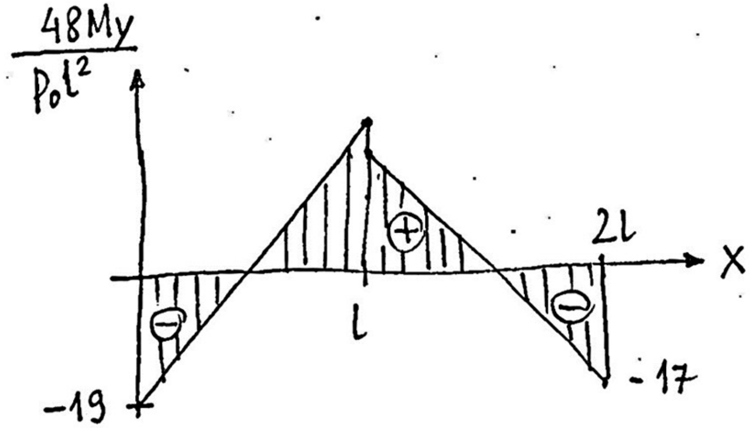
$$F = p_0 \cdot L \quad , \quad M = \frac{p_0 l^2}{6}$$

$$M_{y_1}(0) = -\frac{1}{4}\left(p_0 l^2 + \frac{5}{12} p_0 l^2 + \frac{p_0 l^2}{6}\right) = -\frac{19}{48} p_0 l^2$$

$$M_{y_1}(l) = \frac{1}{4}\left(p_0 l^2 + \frac{1}{3} p_0 l^2 + \frac{1}{3} p_0 l^2\right) = \frac{20}{48} p_0 l^2$$

$$M_{y_2}(0) = \frac{1}{4}\left(p_0 l^2 + \frac{2}{3} p_0 l^2 - \frac{1}{3} p_0 l^2\right) = \frac{16}{48} p_0 l^2$$

$$M_{y_2}(l) = -\frac{1}{4}\left(p_0 l^2 + \frac{7}{12} p_0 l^2 - \frac{p_0 l^2}{6}\right) = -\frac{17}{48} p_0 l^2$$



NORMAL STRESS